Remarks on the sequential effect algebras*

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Abstract

In this paper, first, we answer affirmatively an open problem which was presented in 2005 by professor Gudder on the sub-sequential effect algebras. That is, we prove that if $(E, 0, 1, \oplus, \circ)$ is a sequential effect algebra and A is a commutative subset of E, then the sub-sequential effect algebra $\overline{A}$ generated by A is also commutative. Next, we also study the following uniqueness problem: If $na = nb = c$ for some positive integer $n \geq 2$, then under what conditions $a = b$ hold? We prove that if c is a sharp element of E and $a|b$, then $a = b$. We give also two examples to show that neither of the above two conditions can be discarded.

Key Words. Sub-sequential effect algebras, commutative, uniqueness.

1. Introduction

Effect algebra is an important logic model for studying quantum effects or observations which may be *fuzzy* or *unsharp* (see [1]), to be precise, an effect algebra is a system $(E, 0, 1, \oplus)$, where 0 and 1 are distinct elements of E and $\oplus$ is a partial binary operation on E satisfying:

(EA1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $b \oplus a = a \oplus b$.

*This project is supported by Natural Science Found of China (10771191 and 10471124).

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(EA2) If $a \oplus (b \oplus c)$ is defined, then $(a \oplus b) \oplus c$ is defined and

$$(a \oplus b) \oplus c = a \oplus (b \oplus c).$$

(EA3) For each $a \in E$, there exists a unique element $b \in E$ such that $a \oplus b = 1$.

(EA4) If $a \oplus 1$ is defined, then $a = 0$.

In an effect algebra $(E, 0, 1, \oplus)$, if $a \oplus b$ is defined, we write $a \perp b$. For each $a \in E$, it follows from (EA3) that there exists a unique element $b \in E$ such that $a \oplus b = 1$, we denote b by a'. Let $a, b \in E$, if there exists an element $c \in E$ such that $a \perp c$ and $a \oplus c = b$, then we say that $a \leq b$ and write $c = b \ominus a$. It follows from [1] that $\leq$ is a partial order of $(E, 0, 1, \oplus)$ and satisfies that for each $a \in E$, $0 \leq a \leq 1$, $a \perp b$ if and only if $a \leq b'$.

Let $(E, 0, 1, \oplus)$ be an effect algebra and $a \in E$. If $a \wedge a' = 0$, then a is said to be a *sharp element* of E. The set $E_s = \{x \in E |\ x \wedge x' = 0\}$ is called the set of all sharp elements of E (see [2-3]).

As we knew, two measurements a and b cannot be performed simultaneously in general, so they are frequently executed sequentially ([4]). We denote by $a \circ b$ a sequential measurement in which a is performed first and b second and call $a \circ b$ a *sequential product* of a and b. Thus, it is an important and interesting project to study effect algebras which have a sequential product $\circ$ with some nature properties. To be precise:

A *sequential effect algebra* (SEA) is an effect algebra $(E, 0, 1, \oplus)$ and another binary operation $\circ$ defined on $(E, 0, 1, \oplus)$ satisfying [5]:

(SEA1) The map $b \mapsto a \circ b$ is additive for each $a \in E$, that is, if $b \perp c$, then $a \circ b \perp a \circ c$ and $a \circ (b \oplus c) = a \circ b \oplus a \circ c$.

(SEA2) $1 \circ a = a$ for each $a \in E$.

(SEA3) If $a \circ b = 0$, then $a \circ b = b \circ a$.

(SEA4) If $a \circ b = b \circ a$, then $a \circ b' = b' \circ a$ and for each $c \in E$, $a \circ (b \circ c) = (a \circ b) \circ c$.

(SEA5) If $c \circ a = a \circ c$ and $c \circ b = b \circ c$, then $c \circ (a \circ b) = (a \circ b) \circ c$ and $c \circ (a \oplus b) = (a \oplus b) \circ c$ whenever $a \perp b$.

Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra. If $a, b \in E$ and $a \circ b = b \circ a$, then we say a and b is *sequentially independent* and denoted by $a|b$.

Lemma 1 ([1, 5]). If $(E, 0, 1, \oplus, \circ)$ is a sequential effect algebra and $a, b, c \in E$, then

(1) $a \perp b$, $a \perp c$ and $a \oplus b = a \oplus c$ implies that $b = c$.

(2) $a \in E_s$ if and only if $a \circ a = a$.

(3) If $c \in E_s$, then $a \leq c$ if and only if $a = a \circ c = c \circ a$.

2. Sub-sequential effect algebra generated by a subset

Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra and F a nonempty subset of E. We call F a *sub-sequential effect algebra* of $(E, 0, 1, \oplus, \circ)$ if $0, 1 \in F$ and $(F, 0, 1, \oplus, \circ)$ itself is a sequential effect algebra. From the definition of sub-sequential effect algebra, it is easy to see that a nonempty subset F of $(E, 0, 1, \oplus, \circ)$ is a sub-sequential effect algebra if and only if F is closed under all the three operations $\oplus$, $\circ$ and $'$. Moreover, if A is a nonempty subset of E, it is easy to see that there exists a smallest sub-sequential effect algebra $\overline{A}$ of E which contains A (That is, the intersection of all sub-sequential effect algebras containing A). We call $\overline{A}$ *the sub-sequential effect algebra generated by* A. In 2005, Professor Gudder presented the following open problem (see [6, Problem 17]):

Problem 1. If $(E, 0, 1, \oplus, \circ)$ is a sequential effect algebra and A a commutative subset of E (That is, $a|b$ for all $a, b \in A$), is $\overline{A}$ commutative ?

In this paper, we answer the problem affirmatively. That is:

Theorem 1. Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra and A a commutative subset of $(E, 0, 1, \oplus, \circ)$. Then $\overline{A}$ is also commutative.

Proof. Let $\bigwedge = \{F |\ F$ *be a* commutative subset of E containing $A\}$. We order $\bigwedge$ by including. Using Zorn's Lemma, it is easy to see that there exists a maximal element F_0 in $\bigwedge$. That is, F_0 is a maximal commutative subset of E containing A.

We now prove that F_0 is a sub-sequential effect algebra of E:

If $a \in F_0$, then for each $c \in F_0$, $c|a$, so $c|a'$ by (SEA4). By maximality, we have $a' \in F_0$.

If $a, b \in F_0$, then for each $c \in F_0$, $c|a$, $c|b$, so $c|(a \circ b)$ by (SEA5). By maximality, we have $(a \circ b) \in F_0$.

If $a, b \in F_0$ and $a \perp b$, then for each $c \in F_0$, $c|a$, $c|b$, so $c|(a \oplus b)$ by (SEA5). By maximality, we have $(a \oplus b) \in F_0$.

So F_0 is closed under all the three operations $\oplus$, $\circ$ and $'$.

Thus, F_0 is a sub-sequential effect algebra of $(E, 0, 1, \oplus, \circ)$ containing A. Since $\overline{A}$ is the smallest sub-sequential effect algebra of $(E, 0, 1, \oplus, \circ)$ containing A, we have $\overline{A} \subseteq F_0$ and $\overline{A}$ is also commutative.

Moreover, for general subset A of E, we can describe the structure of $\overline{A}$, that is

Theorem 2. Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra and A a subset of E. If we denote

$A_1 = A \bigcup (\bigcup\limits_{a \in A} a') \bigcup (\bigcup\limits_{a,b \in A} a \circ b) \bigcup (\bigcup\limits_{a,b \in A \ and \ a \perp b} a \oplus b)$,

$A_2 = A_1 \bigcup (\bigcup\limits_{a \in A_1} a') \bigcup (\bigcup\limits_{a,b \in A_1} a \circ b) \bigcup (\bigcup\limits_{a,b \in A_1 \ and \ a \perp b} a \oplus b)$,

$\cdots$

$A_n = A_{n-1} \bigcup (\bigcup\limits_{a \in A_{n-1}} a') \bigcup (\bigcup\limits_{a,b \in A_{n-1}} a \circ b) \bigcup (\bigcup\limits_{a,b \in A_{n-1} \ and \ a \perp b} a \oplus b)$,

$\cdots$

$\Gamma = \bigcup\limits_{n=1}^{\infty} A_n$.

Then $\overline{A} = \Gamma$.

Proof. First we prove that Γ is a sub-sequential effect algebra of $(E, 0, 1, \oplus, \circ)$.

If $a \in \Gamma$, then $a \in A_n$ for some n, so $a' \in A_{n+1} \subseteq \Gamma$.

If $a, b \in \Gamma$, then $a, b \in A_n$ for some n, so $(a \circ b) \in A_{n+1} \subseteq \Gamma$.

If $a, b \in \Gamma$ and $a \perp b$, then $a, b \in A_n$ for some n, so $(a \oplus b) \in A_{n+1} \subseteq \Gamma$.

Thus, Γ is closed under all the three operations $\oplus$, $\circ$ and $'$. So Γ is a sub-sequential effect algebra of $(E, 0, 1, \oplus, \circ)$.

Of course $A \subseteq \Gamma$. Since $\overline{A}$ is the smallest sub-sequential effect algebra of $(E, 0, 1, \oplus, \circ)$ containing A, we have $\overline{A} \subseteq \Gamma$. On the other hand, by induction, it is easy to see that $A_n \subseteq \overline{A}$ for all n. Thus $\Gamma \subseteq \overline{A}$. So $\Gamma = \overline{A}$.

Note that by using Theorem 2 we can also answer professor Gudder's problem by a constructive way, we omit the process.

3. An addition property of sequential effect algebras

Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra, $a, b \in E$. If $\underbrace{a \oplus a \cdots \oplus a}_{\textit{the number is } n}$ is defined, we denote it by na. Now, we are interested in the following uniqueness problem: If for some positive integer $n_0 \geq 2$, $n_0 a = n_0 b$, then under what conditions $a = b$ hold? We have

Theorem 3. Let $(E, 0, 1, \oplus, \circ)$ be a sequential effect algebra, $a, b \in E$ and for some positive integer $n_0 \geq 2$, $n_0 a = n_0 b = c$. If $c \in E_s$ and $a|b$, then $a = b$.

Proof. Since $a \leq c$, by Lemma 1, $a = a \circ c$, similarly $b = b \circ c$.

By (SEA1), we have $a \circ c = a \circ (n_0 b) = n_0(a \circ b)$, $b \circ c = b \circ (n_0 a) = n_0(b \circ a)$.

Note that $a|b$, so $a \circ b = b \circ a$ and $a \circ c = b \circ c$. Thus $a = b$.

Now, we show that neither of the two conditions in Theorem 3 can be discarded.

Example 1. Let $I_1 = [0, 1]$, $I_2 = [0, 1]$, $E = HS(I_1, I_2)$ be the horizontal sum of I_1, I_2 (see [5, Section 8, the Example in P_{109}]). For each $t \in [0, 1]$, if it is in I_1, we denote it by $\hat{t}$; if it is in I_2, we denote it by $\check{t}$. Let $a = \frac{\hat{1}}{n_0}$, $b = \frac{\check{1}}{n_0}$. Then $n_0 a = 1 = n_0 b$, $1 \in E_s$, $a \neq b$, $a \circ b \neq b \circ a$. So the condition $a|b$ in Theorem 3 can not be discarded.

Example 2. Let $\mathbf{N}$ be the nonnegative integer set, n_0 be a positive integer and $n_0 \geq 2$, $E_0 = \{0, 1, a_{n,m}, b_{n,m} |\ n, m \in \mathbf{N},\ n_0 - 1 \geq m,\ n^2 + m^2 \neq 0\}$.

First, we define a partial binary operation $\oplus$ on E_0 as follows (when we write $x \oplus y = z$, we always mean $x \oplus y = z = y \oplus x$):

For each $x \in E_0$, $0 \oplus x = x$,

$$a_{n,m} \oplus a_{r,s} = \begin{cases} a_{n+r,m+s}\ , & if\ m + s < n_0; \\ a_{n+r+n_0,m+s-n_0}\ , & if\ m + s \geq n_0. \end{cases}$$

$$a_{n,m} \oplus b_{r,s} = \begin{cases} b_{r-n,s-m} \ , & if \ n \leq r, \ m \leq s, \ (r-n)^2 + (s-m)^2 \neq 0; \\ 1 \ , & if \ n = r, \ m = s; \\ b_{r-n-n_0,s-m+n_0} \ , & if \ n + n_0 \leq r, \ m > s. \end{cases}$$

No other $\oplus$ operation is defined.

Next, we define a binary operation $\circ$ on E_0 as follows (when we write $x \circ y = z$, we always mean $x \circ y = z = y \circ x$):

For each $x \in E_0$, $0 \circ x = 0$, $1 \circ x = x$,

$a_{n,m} \circ a_{r,s} = 0$, $a_{n,m} \circ b_{r,s} = a_{n,m}$,

$$b_{n,m} \circ b_{r,s} = \begin{cases} b_{n+r,m+s} \ , & if \ m + s < n_0; \\ b_{n+r+n_0,m+s-n_0} \ , & if \ m + s \geq n_0. \end{cases}$$

Now, we prove that E_0 is a sequential effect algebra.

In fact, (EA1) and (EA4) are trivial.

We verify (EA2), for simplicity, we omit the trivial cases about 0,1:

$a_{k,j} \oplus (a_{n,m} \oplus a_{r,s}) = (a_{k,j} \oplus a_{n,m}) \oplus a_{r,s}$

$$= \begin{cases} a_{k+r+n,s+j+m} \ , & if \ s + j + m < n_0; \\ a_{k+r+n+n_0,s+j+m-n_0} \ , & if \ n_0 \leq s + j + m < 2n_0; \\ a_{k+r+n+2n_0,s+j+m-2n_0} \ , & if \ s + j + m \geq 2n_0. \end{cases}$$

Each $a_{k,j} \oplus (a_{n,m} \oplus b_{r,s})$ or $(a_{k,j} \oplus a_{n,m}) \oplus b_{r,s}$ is defined if and only if one of the following four conditions is satisfied, at this case,

$a_{k,j} \oplus (a_{n,m} \oplus b_{r,s}) = (a_{k,j} \oplus a_{n,m}) \oplus b_{r,s}$

$$= \begin{cases} b_{r-k-n,s-j-m} \ , & if \ k + n \leq r, \ j + m \leq s, \ (r-k-n)^2 + (s-j-m)^2 \neq 0; \\ b_{r-k-n-n_0,s-j-m+n_0} \ , & if \ k + n + n_0 \leq r, \ s < j + m \leq n_0 + s, \\ & \qquad (r-k-n-n_0)^2 + (s-j-m+n_0)^2 \neq 0; \\ b_{r-k-n-2n_0,s-j-m+2n_0} \ , & if \ k + n + 2n_0 \leq r, \ n_0 + s < j + m; \\ 1 \ , & if \ (r-k-n)^2 + (s-j-m)^2 = 0 \ or \\ & \qquad (r-k-n-n_0)^2 + (s-j-m+n_0)^2 = 0. \end{cases}$$

Thus, (EA2) is hold.

(EA3) is clear since $a_{n,m} \oplus b_{n,m} = 1$. Thus, $(E_0, 0, 1, \oplus)$ is an effect algebra.

Moreover, we verify that $(E_0, 0, 1, \oplus, \circ)$ is a sequential effect algebra.

(SEA2) and (SEA3) and (SEA5) are trivial.

We verify (SEA1), for simplicity, we omit the trivial cases about 0,1:

$a_{k,j} \circ (a_{n,m} \oplus a_{r,s}) = a_{k,j} \circ a_{n,m} \oplus a_{k,j} \circ a_{r,s} = 0.$

$$b_{k,j} \circ (a_{n,m} \oplus a_{r,s}) = b_{k,j} \circ a_{n,m} \oplus b_{k,j} \circ a_{r,s} = \begin{cases} a_{n+r,m+s} \ , & if \ m+s < n_0; \\ a_{n+r+n_0,m+s-n_0} \ , & if \ m+s \geq n_0. \end{cases}$$

When $a_{n,m} \oplus b_{r,s}$ is defined,

$a_{k,j} \circ (a_{n,m} \oplus b_{r,s}) = a_{k,j} \circ a_{n,m} \oplus a_{k,j} \circ b_{r,s} = a_{k,j}$,

$b_{k,j} \circ (a_{n,m} \oplus b_{r,s}) = b_{k,j} \circ a_{n,m} \oplus b_{k,j} \circ b_{r,s}$

$$= \begin{cases} b_{r+k-n,s+j-m} \ , & if \ n \leq r, \ m \leq s, \ j+s < n_0+m; \\ b_{r+k-n,s+j-m} \ , & if \ n+n_0 \leq r, \ s < m \leq j+s; \\ b_{r+k-n+n_0,s+j-m-n_0} \ , & if \ n \leq r, \ n_0+m \leq j+s; \\ b_{r+k-n-n_0,s+j-m+n_0} \ , & if \ n+n_0 \leq r, \ j+s < m. \end{cases}$$

Thus, (SEA1) is true.

We verify (SEA4), for simplicity, we omit also the trivial cases about 0,1:

$a_{k,j} \circ (a_{n,m} \circ a_{r,s}) = (a_{k,j} \circ a_{n,m}) \circ a_{r,s} = 0.$

$a_{k,j} \circ (a_{n,m} \circ b_{r,s}) = (a_{k,j} \circ a_{n,m}) \circ b_{r,s} = 0.$

$a_{k,j} \circ (b_{n,m} \circ b_{r,s}) = (a_{k,j} \circ b_{n,m}) \circ b_{r,s} = a_{k,j}.$

$b_{k,j} \circ (b_{n,m} \circ b_{r,s}) = (b_{k,j} \circ b_{n,m}) \circ b_{r,s}$

$$= \begin{cases} b_{k+r+n,s+j+m} \ , & if \ s+j+m < n_0; \\ b_{k+r+n+n_0,s+j+m-n_0} \ , & if \ n_0 \leq s+j+m < 2n_0; \\ b_{k+r+n+2n_0,s+j+m-2n_0} \ , & if \ s+j+m \geq 2n_0. \end{cases}$$

Thus (SEA4) is hold and $(E_0, 0, 1, \oplus, \circ)$ is a sequential effect algebra.

Finally, we show that the condition $c \in E_s$ in Theorem 3 can not be discarded.

Indeed, since $a_{n,0} \oplus a_{r,0} = a_{n+r,0}$, so $n_0 a_{1,0} = a_{n_0,0}$. Note that

$$a_{0,m} \oplus a_{0,s} = \begin{cases} a_{0,m+s}\ , & if\ m+s < n_0; \\ a_{n_0,m+s-n_0}\ , & if\ m+s \geq n_0. \end{cases}$$

Thus, $(n_0-1)a_{0,1} = a_{0,n_0-1}$, $n_0 a_{0,1} = (n_0-1)a_{0,1} \oplus a_{0,1} = a_{0,n_0-1} \oplus a_{0,1} = a_{n_0,0}$, that is, $n_0 a_{1,0} = a_{n_0,0} = n_0 a_{0,1}$. Note that $a_{n_0,0} \circ a_{n_0,0} = 0$, so $a_{n_0,0} \notin (E_0)_s$.

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